

Courant Institute of
Mathematical Sciences

Magneto-Fluid Dynamics Division

Coastal Refraction of VLF Waves

D. R. Dobrott

AEC Research and Development Report

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Abstract

The fundamental solution for VLF waves propagating past a shoreline in the earth-ionosphere duct is obtained in terms of an infinite series using the Wiener-Hopf methods. The effects of the land, sea and ionosphere are given in terms of complex impedance boundary conditions. As a result, each mode in the series is progressively damped. At low frequencies and distances far from the shoreline where a single mode dominates, reflection and transmission coefficients are obtained. When the ionospheric impedance is small and when the source and receiver are sufficiently elevated, the effect of the shoreline is not observable. However, when either the source or receiver is near the earth the portion of the wave reflected is proportional to the difference of sea and land impedances. At slightly higher frequencies, there exists an annular region about the shoreline where two modes propagate over the sea while, due to heavy damping, only one mode propagates over the land. The resulting increase of the field strength as the receiver passes from land, seaward is the "recovery effect."

Table of Contents

Abstract	ii
Acknowledgement	iv
Sections:	
1. Introduction	1
2. Integral Representation of the Fundamental Solution. The Wiener-Hopf Problem	7
3. Decomposition	11
4. Coastal Refraction of VLF Waves	
4.1. Explicit Series Representation of Fundamental Solution	18
4.2. Reflection and Transmission Coefficients	21
4.3. The Recovery Effect	27
References	30
Figure Captions	32

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1. Introduction

The purpose of this paper is to derive and analyze the fundamental solution (Green's function) for very-low-frequency (VLF) two-dimensional electromagnetic waves propagating past a shoreline in the earth-ionosphere wave guide. Although portions of this paper have appeared as a report (Dobrott 1960), continued interest in the subject (Wait 1965) suggests that these results be made more readily available.

The original version, which gives the fundamental solution in terms of an infinite set of spatially damped modes, has been condensed. New results are given on the reflection and transmission coefficients, when only one of these modes propagates, and on the "recovery effect" (Millington 1949), where more than one mode propagates resulting in an increase in the magnitude of the vertical component of the electric field as the wave passes from land to sea. Wiener-Hopf techniques are used to solve the appropriate partial differential equations with mixed boundary conditions. In particular, this paper follows methods due to D.S. Jones as presented by Noble (1958).

The analysis of electromagnetic wave propagation over land-sea boundaries has an old and extensive literature. These papers, including some which utilize Wiener-Hopf methods, are not generally applicable to VLF propagation because of the absence of the ionospheric boundary in their

analysis. A partial list of this indirectly related literature is given at the end of this paper.

Several papers using the Wiener-Hopf technique but not dealing with electromagnetic wave propagation, exhibit mathematical similarities to VLF mode transmission. Heins and Feshbach (1947) studied the coupling of sound waves between two dissimilar acoustical ducts. Heins (1948) extended these results when studying the behavior of water waves in a channel of finite depth incident upon a dock. Similarly, Weitz and Keller (1950,1953) have examined the reflection of water waves from floating ice in water of finite depth. The present paper differs from the above in that all three boundaries, i.e. ionosphere, land and sea, are represented by complex impedances. In the above papers, the equivalent of the boundary impedance corresponding to the ionosphere and sometimes that of one other boundary has been set equal to zero. Such a restriction on the ionospheric boundary is physically unreasonable since the nature of the waves in the earth-ionosphere duct is strongly determined by all the boundaries.

Figure 1 is a schematic of the problem to be considered. The z-axis corresponds to the shore line in a right-hand rectangular coordinate system. The earth surface lies in the x-z plane and the ionosphere is taken to have a sharp boundary at $y = 0$. The earth's magnetic field is assumed parallel to the shore line. The symbol T indicates the source and R is the receiver or field point. The sources and

the fields are assumed to vary in time like $\exp(i\omega t)$. Appropriate to the reduction to a two-dimensional problem, the source position and strength is assumed independent of the z-coordinate.

The surface impedance representing the effect of a sharply bounded ionosphere depends upon the direction of propagation relative to the orientation of the earth's magnetic field. For propagation across the earth's field, Maxwell's equations rigorously decouple into transverse magnetic (TM) and transverse electric (TE) waves. For low frequencies, the ionospheric impedance is given approximately by

$$Z_I = \left[\frac{\omega^2 + \nu^2}{\omega_p^2} \right]^{\frac{1}{2}} \exp \left[\frac{\pi}{2} - \tan^{-1} \left(\frac{\nu}{\omega} \right) \right], \quad (1.1)$$

where

$$Z_I = \begin{cases} - \frac{1}{Z_0} \frac{E_x}{H_z} \Big|_{y=0} & \text{for TM waves} \\ \frac{1}{Z_0} \frac{E_z}{H_x} \Big|_{y=0} & \text{for TE waves} \end{cases} \quad (1.2)$$

where $Z_0 = 377$ ohms is the characteristic impedance of free space, ω_p the plasma frequency and ν the electron collision frequency. A signal frequency ω is considered low when $\omega \ll \omega_c \ll \omega_p$, where ω_c is the electron cyclotron frequency in the earth's field. At higher frequencies the ionospheric

impedance for TE and TM waves are different (see Dobrott 1961); however the fundamental solution may be trivially modified by substitution of the appropriate expression for Z_I in the separate cases.

The relative land and sea impedances are given by (see Kerr 1951)

$$Z_{L,S} = \left[\epsilon_{1L,S}^2 + \left(\frac{\sigma_{L,S}}{\omega \epsilon_0} \right)^2 \right]^{-\frac{1}{2}} \exp \left[\frac{1}{2} \frac{\sigma_{L,S}}{\omega \epsilon_0 \epsilon_{1L,S}} \right], \quad (1.3)$$

where

$$Z_{L,S} = \begin{cases} \frac{1}{Z_0} \frac{E_x}{H_z} \Big|_{y=0} & \text{for TM waves} \\ -\frac{1}{Z_0} \frac{E_z}{H_x} \Big|_{y=0} & \text{for TE waves} \end{cases} \quad (1.4)$$

where $\epsilon_{1L,S}$ is the relative dielectric constant and $\sigma_{L,S}$ is the conductivity for land and sea respectively. All impedance magnitudes are assumed small, in the order $|Z_S| \ll |Z_I| \ll |Z_L| \ll 1$.

The z components of the electric and magnetic fields produced by a current distribution $j(x,y) \exp(i\omega t)$ are given in terms of the fundamental solution $g(x_0, y_0 | x, y)$ as

$$H_z(x, y, t) = \exp(i\omega t) \oint\!\!\!\oint_{\text{source}} j(x_0, y_0) x \nabla_0 \cdot \hat{i}_z g(x_0, y_0 | x, y) dx_0 dy_0 \quad (1.5)$$

where $\hat{i}_z$ is the unit vector in the z-direction and

$$E_z(x,y,t) = -i\omega\mu_0 \exp(i\omega t) \oint\!\!\!\oint_{\text{source}} \underline{j}(x_0, y_0) \cdot \hat{i}_z g(x_0, y_0 | x, y) dx_0 dy_0 \quad (1.6)$$

where the fundamental solution $g(x_0, y_0 | x, y)$ satisfies the equation

$$\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} + k^2 g = 0 \quad (1.7)$$

for all $x \neq x_0$ and $y \neq y_0$, the condition

$$\lim_{\epsilon \rightarrow 0} \left\{ \int_{y_0 - \epsilon}^{y_0 + \epsilon} dy \left[\left. \frac{\partial g}{\partial x} \right|_{x_0 + \epsilon} - \left. \frac{\partial g}{\partial x} \right|_{x_0 - \epsilon} \right] + \int_{x_0 - \epsilon}^{x_0 + \epsilon} dx \left[\left. \frac{\partial g}{\partial y} \right|_{y_0 + \epsilon} - \left. \frac{\partial g}{\partial y} \right|_{y_0 - \epsilon} \right] \right\} = -1 \quad (1.8)$$

and the boundary conditions

$$\frac{\partial g}{\partial y} + ikZ_I g = 0 \quad \text{when } y = b, \quad -\infty < x < \infty \quad (1.9)$$

$$\frac{\partial g}{\partial y} - ikZ_L g = 0 \quad \text{when } y = 0, \quad 0 < x < \infty \quad (1.10)$$

$$\frac{\partial g}{\partial y} - ikZ_S g = 0 \quad \text{when } y = 0, \quad -\infty < x < 0. \quad (1.11)$$

where $k = \omega/c$ and c is the velocity of light in a vacuum.

For the purposes of uniqueness, it is necessary to specify conditions at the shoreline and at infinity. At the shoreline, electromagnetic power is neither absorbed nor radiated. This is equivalent to the statement,

$$\lim_{\rho \rightarrow 0} \int_0^{\pi} g^*(x_0, y_0 | \rho \cos \theta, \rho \sin \theta) \frac{\partial g(x_0, y_0 | \rho \cos \theta, \rho \sin \theta)}{\partial \rho} \rho d\theta = 0 \quad (1.12)$$

where g^* denotes the complex conjugate of g , $x = \rho \cos \theta$, $y = \rho \sin \theta$ and $0 \leq \theta \leq \pi$. Similarly, since the duct boundaries are lossy, no power will reach infinity. This statement is the same as saying

$$\lim_{|x| \rightarrow \infty} \int_0^b g^*(x_0, y_0 | x, y) \frac{\partial g(x_0, y_0 | x, y)}{\partial x} dy = 0. \quad (1.13)$$

The z components suffice to characterize the TM and TE modes for this problem. The other components are determined, in principle, by Maxwell's equations. Physically, $\mu_0 g(x_0, y_0 | x, y) \exp(i\omega t)$ is the two-dimensional vector potential at (x, y) produced by a dipole source at (x_0, y_0) .

In Section 2, the integral representation for the fundamental solution is given in terms of an integral involving the Fourier transform in x of g evaluated at $y = 0$. This transformed function is unknown and is determined in an auxiliary problem by Wiener-Hopf techniques (Section 3). An explicit series representation for the fundamental solution is given in Section 4. The particular asymptotic form for the solution well away from the shoreline is used to obtain reflection and transmission coefficients. The recovery effect is obtained by examining the solution in an annulus about the earth-sea juncture for frequencies which permit more than one mode to propagate in the earth-ionosphere duct.

2. Integral Representation of the Fundamental Solution

The Wiener-Hopf Problem

The fundamental solution is obtained by taking the Fourier transform in x of Eq.(1.7) and solving the resulting ordinary differential equation in y which formally satisfies the transformed equivalents of Eqs. (1.8 - 1.9).^{*} The transform is defined for convenience to be decomposed by sum as

$$G(y, \alpha) = G_+(y, \alpha) + G_-(y, \alpha) \quad (2.1)$$

where

$$G_{\pm}(y, \alpha) = \frac{\pm 1}{(2\pi)^{\frac{1}{2}}} \int_0^{\pm \infty} g e^{i\alpha x} dx \quad (2.2)$$

Upon taking the inverse transform, one obtains the representation

$$g(x_0, y_0 | x, y) = g_1(x_0, y_0 | x, y) + \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{-\infty}^{\infty} d\alpha e^{-i\alpha x} \left[\frac{\cosh \gamma(b-y) + ikZ_I \frac{\sinh \gamma(b-y)}{\gamma}}{\cosh \gamma b + ikZ_I \frac{\sinh \gamma b}{\gamma}} \right] G(0, \alpha) \quad (2.3)$$

^{*} The equivalent condition of Eq.(1.8) then becomes

$$\lim_{\epsilon \rightarrow 0} \left\{ \left. \frac{\partial G}{\partial y}(y, \alpha) \right|_{y_0 + \epsilon} - \left. \frac{\partial G}{\partial y}(y, \alpha) \right|_{y_0 - \epsilon} \right\} = - \frac{e^{i\alpha x_0}}{(2\pi)^{\frac{1}{2}}}$$

where

$$g_1(x_0, y_0 | x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{d\alpha e^{-i\alpha(x-x_0)}}{\cosh \gamma b + ikZ_I \frac{\sinh \gamma b}{\gamma}}$$

$$\times \begin{cases} \frac{\sinh \gamma y}{\gamma} \left[\cosh \gamma(b-y_0) + ikZ_I \frac{\sinh \gamma(b-y_0)}{\gamma} \right] & \text{for } 0 \leq y \leq y_0 \\ \frac{\sinh \gamma y_0}{\gamma} \left[\cosh \gamma(b-y) + ikZ_I \frac{\sinh \gamma(b-y)}{\gamma} \right] & \text{for } y_0 \leq y \leq b. \end{cases}$$

(2.4)

and where γ has been defined by

$$\gamma = \sqrt{a^2 - k^2} \quad (2.5)$$

The integrands of Eqs. (2.3-.4) are even functions of γ and hence possess no branch points.

The fundamental solution formally* satisfies the boundary conditions given by Eqs. (1.10-.11) provided the following relation

$$C(\alpha)G_+(0, \alpha) + D(\alpha)G_-(0, \alpha) + E(\alpha) = 0 \quad (2.6)$$

holds for all real α , where

*Once the properties of $G(0, \alpha)$ are determined formally, it will be indicated how Eqs. (1.7-.11) are satisfied rigorously.

$$C(\alpha) = ik(Z_L + Z_I) \cosh \gamma b + \gamma \sinh \gamma b \quad (2.7)$$

$$D(\alpha) = ik(Z_S + Z_I) \cosh \gamma b + \gamma \sinh \gamma b \quad (2.8)$$

and

$$E(\alpha) = - \frac{\exp(i\alpha x_0)}{(2\pi)^{\frac{1}{2}}} \left[\cosh \gamma (b - y_0) + ikZ_I \frac{\sinh \gamma (b - y_0)}{\gamma} \right] \quad (2.9)$$

and where quadratic terms in the impedances have been neglected, i.e., $Z_I Z_L \approx Z_I Z_S \approx 0$. Provided g has a Fourier transform and $ge^{\beta_1 x}$ and $ge^{-\beta_2 x}$ are absolute integrable in the respective ranges $[0, +\infty)$ and $(-\infty, 0]$ for β_1, β_2 positive and real, Eq. (2.6) is analytic in the strip

$$- \min(\beta_1, \operatorname{Im} k_0, \operatorname{Im} h_0) < \operatorname{Im} \alpha < \min(\beta_2, \operatorname{Im} k_0, \operatorname{Im} h_0),$$

where k_0 and h_0 are complex constants corresponding to the respective zeros of $C(\alpha)$ and $D(\alpha)$ which lie closest to the real axis. Further, the functions $G_+(y, \alpha)$ and $G_-(y, \alpha)$ are then analytic in α for α in the respective domain

$$\operatorname{Im} \alpha > -\beta_1 \quad \text{and} \quad \operatorname{Im} \alpha < \beta_2.$$

The essence of the Wiener-Hopf method is to decompose into a product the coefficients of $G_+(0, \alpha)$ and $G_-(0, \alpha)$ in Eq. (2.6) so that

$$C(\alpha) = C_+(\alpha) C_-(\alpha) \quad (2.10)$$

and

$$D(\alpha) = D_+(\alpha)D_-(\alpha) \quad (2.11)$$

The regions of analyticity of $C_+(\alpha)$ and $C_-(\alpha)$ functions are $\text{Im}\alpha > -\text{Im}k_0$ and $\text{Im}\alpha < \text{Im}k_0$ respectively, and those of $D_+(\alpha)$ and $D_-(\alpha)$ are $\text{Im}\alpha > -\text{Im}h_0$ and $\text{Im}\alpha < \text{Im}h_0$ respectively. A second decomposition into a sum is performed so that

$$\frac{E(\alpha)}{D_+(\alpha)C_-(\alpha)} = H_+(\alpha) + H_-(\alpha) \quad (2.12)$$

where $H_+(\alpha)$ and $H_-(\alpha)$ are analytic in the respective domains $\text{Im}\alpha > -\min(h_0, p_0)$ and $\text{Im}\alpha < \min(k_0, p_0)$, where p_0 is that singularity of $E(\alpha)$ which lies closest to the real α -axis. Equation (2.6) may then be written as

$$\frac{C_+(\alpha)}{D_+(\alpha)} G_+(0, \alpha) + H_+(\alpha) = - \frac{D_-(\alpha)}{C_-(\alpha)} G_-(0, \alpha) - H_-(\alpha). \quad (2.13)$$

The functions $H_{\pm}(\alpha)$ are bounded for real α and are $O(\alpha^{-1})$ as $|\alpha| \rightarrow \infty$ and the ratios $C_+(\alpha)/D_+(\alpha)$ and $D_-(\alpha)/C_-(\alpha)$ are also bounded for real α and are $O(1)$ as $|\alpha| \rightarrow \infty$. (Both of these assertions will be verified in the next section.) Further, provided the above assumptions on g hold, the Riemann-Lebesgue lemma implies that $G_{\pm}(0, \alpha)$ vanish in a strip about $\text{Im}\alpha = 0$ as $|\alpha| \rightarrow \infty$. This strip is common to both sides of Eq. (2.13), which are equal to some function analytic in the strip. This function may be continued analytically throughout the entire α -plane, but

because of the above described asymptotic behavior, the resulting entire function must be zero. Therefore one obtains uniquely

$$G(0, \alpha) = - \frac{E(\alpha)}{C(\alpha)} - ik(Z_L - Z_S) \frac{\cosh \gamma b}{C(\alpha)} \frac{C_-(\alpha)H_-(\alpha)}{D_-(\alpha)} \quad (2.14)$$

The first term in Eq. (2.14) is $O(\alpha^{-1})$ and the second term is $O(\alpha^{-2})$ as $|\alpha| \rightarrow \infty$ and $G(0, \alpha)$ is bounded and continuous for all real α , provided the above assertions about $C_-(\alpha)/D_-(\alpha)$ and $H_-(\alpha)$ can be demonstrated. It remains to decompose $C(\alpha)$, $D(\alpha)$ and $E(\alpha)/C_-(\alpha)D_+(\alpha)$ and show that the assertions may be verified.

3. Decomposition

The decomposition into a product given formally by Eqs.(2.10) and (2.11) and then by sum given by Eq.(2.12) is the crucial part of any problem solved by the Wiener-Hopf technique. The procedure is so well-known that these factors can frequently be written down by inspection. In essence, Eqs. (2.7) and (2.8) are expressed as infinite products and Eq. (2.12) decomposed by an application of the Cauchy integral theorem. By standard methods [see for example Knopp (1957)] one obtains

$$C_{\pm}(\alpha) = \frac{\lambda_0 \pi}{\sqrt{b}} \prod_{n=0}^{\infty} \left[\left\{ 1 - \left(\frac{kb}{\lambda_n \pi} \right)^2 \right\}^{\frac{1}{2}} \mp \frac{i\alpha b}{\lambda_n \pi} \right] e^{\pm i\alpha b/n\pi} \quad (3.1)$$

where λ_n satisfies the equation

$$\lambda_n \pi \tan \lambda_n \pi = ikb(Z_I + Z_L), \quad (3.2)$$

and by successive approximations using the fact that

$|Z_I + Z_L| \ll 1$, one obtains

$$\lambda_0 = \frac{1}{\pi} [ikb(Z_I + Z_L)]^{\frac{1}{2}} \quad (3.3)$$

and

$$\lambda_n = n + \frac{\lambda_0^2}{n}, \quad n = 1, 2, 3, \dots \quad (3.4)$$

The other coefficient is decomposed as

$$D_{\pm}(\alpha) = \frac{\sigma_0 \pi}{\sqrt{b}} \prod_{n=0}^{\infty} \left[\left\{ 1 - \left(\frac{kb}{\sigma_n \pi} \right)^2 \right\}^{\frac{1}{2}} \mp \frac{i\alpha b}{\sigma_n \pi} \right] e^{\pm i\alpha b/n\pi} \quad (3.5)$$

where σ_n satisfies Eq. (3.2) with Z_L replaced by Z_S .

Similar expressions to Eq. (3.3) and (3.4) may all be written for σ_n .

It follows from Eqs. (3.1) and (3.5) that

$$\lim_{\alpha \rightarrow \infty} \frac{C_+(\alpha)}{D_+(\alpha)} = \lim_{\alpha \rightarrow \infty} \frac{C_-(\alpha)}{D_-(\alpha)} = \prod_{n=1}^{\infty} \left(\frac{\sigma_n}{\lambda_n} \right). \quad (3.6)$$

Hence $C_+(\alpha)/D_+(\alpha)$ and $D_-(\alpha)/C_-(\alpha)$ are $O(1)$ as was asserted.

The zeros of Eqs. (3.1) and (3.5) occur at $\pm k_n$ and $\pm h_n$

respectively, where

$$k_n^2 = k^2 - \left(\frac{\lambda_n \pi}{b}\right)^2 \quad \text{and} \quad h_n^2 = k^2 - \left(\frac{\sigma_n \pi}{b}\right)^2 \quad (3.7)$$

These roots lie in the second and fourth quadrants of the α -plane and off the real axis as shown in Figure 2. The condition that h_0 lies closer to the real axis than k_0 is

$$|\Omega_L| - |\Omega_S| > \text{Im}(\Omega_L - \Omega_S)$$

where

$$\Omega_L = Z_I + Z_L + ikb \quad (3.8)$$

and

$$\Omega_S = Z_I + Z_S + ikb \quad (3.9)$$

A sufficient condition that h_0 be closer to the real α -axis than k_0 may be found from the above inequality to be

$$kb + \text{Re}(Z_I + Z_L) > |\Omega_S|$$

which is generally satisfied when the land impedance is much greater than the sea and ionosphere impedance.

It has already been mentioned that $E(\alpha)/D_+(\alpha)C_-(\alpha)$ is analytic in the strip $-\min(\text{Im } h_0, \text{Im } p_0) < \text{Im } \alpha < \min(\text{Im } k_0, \text{Im } p_0)$. It is therefore possible to express this function as the Cauchy integral

$$\frac{E(\alpha)}{D_+(\alpha)C_-(\alpha)} = \frac{1}{2\pi i} \oint_L \frac{E(z)/D_+(z) C_-(z)}{z-\alpha} dz \quad (3.10)$$

where L is a rectangular path shown in Fig. 3. Provided $y_0 \neq 0$, the contributions to the integral from the ends of the rectangle vanish exponentially as these ends are taken to infinity. From the remaining portions of the path, the $H_{\pm}(\alpha)$ functions are obtained

$$H_+(\alpha) = \frac{1}{2\pi i} \int_{ic-\infty}^{ic+\infty} \frac{E(z)/D_+(z) C_-(z)}{z-\alpha} dz \quad (3.11)$$

and

$$H_-(\alpha) = - \frac{1}{2\pi i} \int_{id-\infty}^{id+\infty} \frac{E(z)/D_+(z) C_-(z)}{z-\alpha} dz \quad (3.12)$$

where $-\min(\text{Im } h_0, \text{Im } p_0) < c < \text{Im } \alpha < d < \min(\text{Im } k_0, \text{Im } p_0)$. To evaluate Eqs. (3.11- .12) the line integrals are displaced into the upper-half z -plane for $x_0 > 0$ and into the lower-half plane for $x_0 < 0$. As the paths are displaced to infinity in either half-plane, contributions from poles are accrued and the remaining contribution to the displaced path becomes exponentially small. Note from the above that the $H_{\pm}(\alpha)$ functions are $O(\alpha^{-1})$ as $|\alpha| \rightarrow \infty$, as was asserted earlier.

In what follows, x_0 is taken to be positive, i.e., the transmitter is over the land portion of the duct. Then upon displacing the integration path into the upper-half z -plane as shown in Fig. 4, one obtains for $H_-(\alpha)$

$$H_-(\alpha) = - \frac{1}{(2\pi)^{\frac{1}{2}}} \sum_{n=0}^{\infty} \frac{\exp(ik_n x_0)}{(\alpha - k_n) k_n b} \left(\frac{k_n + k_0}{k_n + h_0} \right) \prod_{\ell=1}^{\infty} \left(\frac{\sigma_{\ell}}{\lambda_{\ell}} \right) \frac{k_n + k_{\ell}}{k_n + h_{\ell}} \\ \times \frac{\cos \frac{\lambda_n \pi}{b} y_0 + \frac{ikbZ_L}{\lambda_n \pi} \sin \frac{\lambda_n \pi}{b} y_0}{1 + (\lambda_0/\lambda_n)^2} \quad (3.13)$$

where quadratic terms in the impedances have been neglected. When $\alpha \neq k_n$ and $y_0 \neq 0$, the above infinite series is uniformly convergent. The $H_+(\alpha)$ function is then found directly by means of Eq. (2.12).

Therefore with Eqs. (3.1), (3.5) and (3.13) it is possible to obtain an explicit series representation for the fundamental solution of all x in $(-\infty, \infty)$ and all y in $[0, b]$ by residue calculus. The residues are due to poles at $\pm k_n$, $\pm h_n$ and $\pm p_n$, $n = 0, 1, 2, 3, \dots$, where p_n is given in terms of the relation

$$p_n^2 = k^2 - \left(\frac{\mu_n \pi}{b} \right)^2 \quad (3.14)$$

where, in turn, the μ_n are the zeros of $E(\alpha)|_{y_0} = 0$ found from the equation

$$\mu_n \pi \cot \mu_n \pi = -ikbZ_I \quad (3.15)$$

which has approximate solutions for $|Z_I| \ll 1$,

$$\mu_n = n + \frac{1}{2} + \frac{ikbZ_I}{(n+\frac{1}{2})\pi^2} . \quad (3.16)$$

Using arguments similar to those given above, it can be shown that generally $|\text{Im } h_0| < |\text{Im } p_0|$ whenever $|Z_S| \ll |Z_I|$.

All the formal results are now available to allow one to show that Eq. (2.3) does satisfy the conditions given by Eqs. (1.7-.13). The integrands of Eq. (2.4) and that given explicitly in Eq. (2.3) are bounded and continuous for all real α and vanish exponentially for all x and y in the respective ranges $-\infty < x < \infty$ and $0 \leq y \leq b$ except for a neighborhood about (x_0, y_0) in the former case and a neighborhood about the origin $(0,0)$ in the latter case. Then exclusive of these neighborhoods, the integrals converge uniformly, limits may be taken under the integral and it is routine to show that the fundamental solution given by Eq. (2.3) indeed satisfies Eqs. (1.7-.11).

In the neighborhood of the shoreline the fundamental solution is given by

$$g(x_0, y_0 | \rho \cos \theta, \rho \sin \theta) = \frac{k(Z_L - Z_S)M_1}{\pi} \rho \cos \theta \log \rho + \text{a bounded function} \quad (3.17)$$

where M_1 is a complex constant depending upon x_0 , y_0 and the physical parameters of the problem. From Eq. (3.17) it is seen that the condition given by Eq. (1.12) is satisfied.

Equation (3.17) is obtained by separating the integration into two ranges: a range bounded away from $\alpha = 0$, from which the logarithmic term is obtained, and the complement of this range. The integral over this latter range is bounded. This coupling between the behavior at the discontinuity and large values of α is a well-known phenomenon (See for example Noble 1958). Generally, in diffraction problems, the behavior at the discontinuity is specified and the corresponding condition for large α is used to separate the two sides of Eq. (2.13).

For $|x| \gg |x_0| \gg 0$, the modulus of the dominant term in the asymptotic form of the fundamental solution is less than $M_2 \exp[-|\min(\text{Im}k_0, \text{Im}p_0)|x]$ for x positive and less than $M_3 \exp[\min(|\text{Im}h_0, \text{Im}p_0|)|x]$ for x negative, where M_2 and M_3 are bounded, positive and real constants. Hence it can be seen that Eq. (1.13) is satisfied. In a similar way, one may recover the fact that the fundamental solution near (x_0, y_0) is $-(1/\pi) \log[(x-x_0)^2 + (y-y_0)^2]^{\frac{1}{2}}$ plus a bounded function which is the standard source term in two-dimensional problems. Derivatives of the fundamental solution near (x_0, y_0) vary as the inverse distance from the source plus logarithmic terms.

4. Coastal Refraction of VLF Wave

4.1 Explicit Series Representation of Fundamental Solution

Let $J(y, \alpha)$ be defined as

$$J(y, \alpha) = \cosh \gamma(b-y) + ikbZ_I \frac{\sinh \gamma(b-y)}{\gamma b} . \quad (4.1)$$

Note that the zeros of $J(0, \alpha)$ are those of $E(\alpha)|_{y_0=0}$ which are determined from Eqs. (3.14) and (3.15). Using the above equation, Eq. (2.3) may be rewritten as

$$\begin{aligned} g(x_0, y_0 | x, y) = & \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{d\alpha e^{-i\alpha(x-x_0)}}{J(0, \alpha)} \begin{cases} \frac{\sinh \gamma y}{\gamma} J(y_0, \alpha) & \text{for } 0 \leq y \leq y_0 \\ \frac{\sinh \gamma y_0}{\gamma} J(y, \alpha) & \text{for } y_0 \leq y \leq b \end{cases} \\ & + \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{-\infty}^{\infty} \frac{d\alpha e^{-i\alpha x} J(y, \alpha)}{J(0, \alpha) C(\alpha)} \left[\frac{e^{i\alpha x_0}}{(2\pi)^{\frac{1}{2}}} J(y_0, \alpha) + D_+(\alpha) C_-(\alpha) H_-(\alpha) \right] \\ & - \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{-\infty}^{\infty} \frac{d\alpha e^{-i\alpha x} J(y, \alpha)}{J(0, \alpha) D(\alpha)} [D_+(\alpha) C_-(\alpha) H_-(\alpha)] . \end{aligned} \quad (4.2)$$

The above integrals may be evaluated by residue calculus.

All integrands in Eq. (4.2) have poles at $\pm p_n$. The integrand of the second integral has poles at $\pm k_n$ and that of the third integral has poles at $\pm h_n$ ($n = 0, 1, 2, \dots$). Thus the series

representation for Eq. (4.3) is

$$\begin{aligned}
g(x_0, y_0 | x, y) &= g_2(x_0, y_0 | x, y) \\
&+ (2\pi)^{\frac{1}{2}} \sum_{n=0}^{\infty} \frac{e^{-ip_n x + i\pi/2} J(y, p_n) H_-(p_n)}{\partial J(0, \alpha) / \partial \alpha |_{\alpha=p_n}} \left[\frac{D_+(p_n)}{C_+(p_n)} - \frac{C_-(p_n)}{D_-(p_n)} \right] \\
&+ (2\pi)^{\frac{1}{2}} \sum_{n=0}^{\infty} \frac{e^{-ih_n x - i\pi/2} J(y, h_n) D_+(h_n) C_-(h_n) H_-(h_n)}{J(0, h_n) D'(h_n)}
\end{aligned} \tag{4.4}$$

for $x < 0$,

$$\begin{aligned}
g(x_0, y_0 | x, y) &= g_2(x_0, y_0 | x, y) \\
&+ (2\pi)^{\frac{1}{2}} \sum_{n=0}^{\infty} \frac{e^{+ip_n x + i\pi/2} J(y, p_n) H_-(-p_n)}{\partial J(0, \alpha) / \partial \alpha |_{\alpha=p_n}} \left[\frac{D_-(p_n)}{C_-(p_n)} - \frac{C_+(p_n)}{D_+(p_n)} \right] \\
&+ \sum_{n=0}^{\infty} \frac{e^{-ik_n(x-x_0) + i\pi/2} J(y, k_n) J(y_0, k_n)}{J(0, k_n) C'(k_n)} \\
&+ (2\pi)^{\frac{1}{2}} \sum_{n=0}^{\infty} \frac{e^{ik_n x + i\pi/2} J(y, k_n) D_-(k_n) C_+(k_n) H_-(-k_n)}{J(0, k_n) C'(k_n)}
\end{aligned} \tag{4.5}$$

for $0 < x < x_0$ and

$$\begin{aligned}
g(x_0, y_0 | x, y) &= g_2(x_0, y_0 | x, y) \\
&+ (2\pi)^{\frac{1}{2}} \sum_{n=0}^{\infty} \frac{e^{ip_n x + i\pi/2} J(y, p_n) H_-(-p_n)}{\partial J(0, \alpha) / \partial \alpha |_{\alpha=p_n}} \left[\frac{D_-(p_n)}{C_-(p_n)} - \frac{C_+(p_n)}{D_+(p_n)} \right] \\
&+ \sum_{n=0}^{\infty} \frac{e^{ik_n(x-x_0)} J(y, k_n) J(y_0, k_n)}{J(0, k_n) C'(k_n)} \\
&+ (2\pi)^{\frac{1}{2}} \sum_{n=0}^{\infty} \frac{e^{ik_n x + i\pi/2} J(y, k_n) D_-(k_n) C_+(k_n) H_-(-k_n)}{J(0, k_n) C'(k_n)}
\end{aligned} \tag{4.6}$$

for $x > x_0$, where

$$\begin{aligned}
g_2(x_0, y_0 | x, y) &= \sum_{n=0}^{\infty} \frac{e^{ip_n |x-x_0| + i\pi/2}}{\partial J(0, \alpha) / \partial \alpha |_{\alpha=p_n}} \left[\frac{J(y, p_n) J(y_0, p_n)}{C(p_n)} \right. \\
&\quad \left. + \begin{cases} \frac{\sin \mu_n \pi y/b}{\mu_n \pi/b} J(y_0, p_n) \\ \text{for } 0 \leq y \leq y_0 \\ \frac{\sin \mu_n \pi y_0/b}{\mu_n \pi/b} J(y, p_n) \\ \text{for } y_0 \leq y \leq b \end{cases} \right]
\end{aligned} \tag{4.7}$$

In writing Eqs. (4.4-7) use has been made of the fact that $C(\alpha)$, $D(\alpha)$, $J(y, \alpha)$ and $J(y_0, \alpha)$ are even functions of α and that $C_{\pm}(\alpha) = C_{\mp}(-\alpha)$ and $D_{\pm}(\alpha) = D_{\mp}(-\alpha)$.

The various terms in the series representation lend themselves readily to physical interpretation. The first two summations in Eqs. (4.4-.6) are the "sky-wave" portion of the solution, i.e. that part which is unobservable if the receiver is on the ground ($y = 0$). The remaining summation in Eq. (4.4) are "ground waves" transmitted beyond the shoreline. The last two summations in Eq. (4.5) are, respectively, ground waves from the source which are incident on the shoreline and ground waves reflected back from the coast.

Waves from the source, but receding away from the shore in the region $x > x_0$, and the ground waves reflected from the coast are given respectively as the last two terms of Eq. (4.6).

4.2. Reflection and Transmission Coefficients

When the receiver and the transmitter are remote from the shoreline and each other, only the $n = 0$ modes of Eqs. (4.4-.6) are observable. To this remote observer, there exists a simple wave incident on, reflected from and transmitted beyond the discontinuity at the shoreline. Weitz and Keller (1953), in a different context, have expressed the situation where a simple wave propagates past a discontinuity in terms of reflection and transmission coefficients.

Consider the source to be a short, two-dimensional dipole at (x_1, y_1) with current moment IL so that the current density is given as

$$J_y(x_o, y_o) = IL\delta(x_o - x_1)\delta(y_o - y_1). \quad (4.8)$$

A vertically oriented antenna at the receiver observes the E_y component of the electric field. For the above source, this field component is given as

$$E_y(x, y, t) = \frac{iIL}{\omega\epsilon_o} e^{i\omega t} \frac{\partial^2}{\partial x_1 \partial x} g(x_1, y_1 | x, y). \quad (4.9)$$

For $|x - x_o|$ large and x large and positive

$$E_y(x, y, t) \sim e^{-i\xi x} + \rho e^{i\xi x} \quad (4.10)$$

and for x large and negative

$$E_y \sim \tau e^{-ih_o x} \quad (4.11)$$

where ξ is either p_o or k_o , which ever has the smallest imaginary part. It is assumed that $\min(\text{Im } p_o, \text{Im } k_o) > \text{Im } h_o$ in any case. The reflection coefficient ρ is defined by

$$\rho = \left. \frac{E_y^{\text{reflected}}}{E_y^{\text{incident}}} \right|_{x=0} \quad (4.12)$$

and the transmission coefficient by

$$\tau = \left. \frac{E_y^{\text{transmitted}}}{E_y^{\text{incident}}} \right|_{x=0} \quad (4.13)$$

If either the transmitter or the receiver are near the ground then only the mode corresponding to $\xi = k_0$ is observable. If $|\text{Im } p_0| > |\text{Im } k_0|$, then $\xi = k_0$ in any case. However, when $|\text{Im } k_0| > |\text{Im } p_0|$ and y and y_0 are not small, $\xi = p_0$. The condition that this latter statement be true is given by

$$|\Omega_L| - |\Omega_I| > \text{Im}(\Omega_L - \Omega_I)$$

where Ω_L and Ω_S are given by Eqs. (3.8) and (3.9) respectively and

$$\Omega_I = 2Z_I + ikb - \frac{i\pi^2}{4kb} . \quad (4.14)$$

A sufficient condition that $\text{Im } k_0 > \text{Im } p_0$ is

$$kb + \text{Re}(Z_L + Z_I) > |\Omega_I|$$

which is usually satisfied when $|Z_L| \gg |Z_I|$ or when $|Z_L| > |Z_I|$ and the earth-ionosphere duct is a quarter wavelength in height.

Because there is an ambiguity as to which mode is propagating, the two cases are discussed separately. For

case I, where $\xi = k_0$ characterizes the dominant mode for $x > 0$, the reflection coefficient ρ_I calculated from Eqs. (4.5) and (4.9) is given as

$$\rho_I = \frac{D(k_0)C_+(k_0)}{J(y_0, k_0)D_+(k_0)} \hat{H}_-(-k_0) \quad (4.15)$$

where $H_-(\alpha) = -\exp(ik_0 x_0) \hat{H}_-(\alpha) / (2\pi)^{\frac{1}{2}}$.

When quadratic terms in the impedances are neglected, ρ_I is given approximately by

$$\rho_I \cong \frac{h_0 - k_0}{h_0 + k_0} \quad (4.16)$$

or since $kb \gg |Z_I + Z_L|$,

$$\rho_I \cong \frac{Z_s - Z_L}{i4kb} \quad (4.17)$$

The transmission coefficient for case I, calculated from Eq. (4.4-.5) and Eq. (4.9) is

$$\tau_I = \frac{h_0 D_+(h_0) C'_-(k_0) C(h_0) \hat{H}_-(h_0) J(0, k_0) J(y, h_0)}{k_0 C_+(h_0) D'_-(h_0) J(y_0, k_0) J(0, h_0) J(y, k_0)} \quad (4.18)$$

which to quadratic terms in the impedances is given as

$$\tau_I = \frac{2h_0}{h_0 + k_0} \quad (4.19)$$

Equations (4.16) and (4.19) are the same form as one classically obtains for a plane wave passing between two dissimilar dielectric media.

For case II, where $\xi = p_o$ characterizes the mode for $x > 0$, the situation is not so simple. Using Eqs. (4.5) and (4.9), one obtains for the reflection coefficient ρ_{II} ,

$$\rho_{II} = \rho_I R(x_o) F_1(p_o, k_o) \quad (4.20)$$

where $R(x_o)$ is the "scale range" defined by

$$R(x_o) = \exp[i(k_o - p_o)x_o] \quad (4.21)$$

and $F_1(p_o, k_o)$ is of polynomial order defined by

$$F_1(p_o, k_o) = \frac{k_o C(p_o) \hat{H}_-(-p_o) J(y_o, k_o) D_+(k_o)}{p_o D(k_o) \hat{H}_-(-k_o) J(y_o, p_o) C_+(k_o)} \left[\frac{D_-(p_o)}{C_-(p_o)} - \frac{C_+(p_o)}{D_+(p_o)} \right] \\ \times \left[1 + \frac{b}{\mu_o \pi} C(p_o) \begin{cases} \frac{\sin \mu_o \pi y / b}{J(y, p_o)} \text{ for } 0 \leq y \leq y_o \\ \frac{\sin \mu_o \pi y_o / b}{J(y_o, p_o)} \text{ for } y_o \leq y \leq b \end{cases} \right]^{-1} \quad (4.22)$$

The transmission coefficient for case II is found from Eqs. (4.4-.5) and Eq. (4.9) to be

$$\tau_{II} = 1 + \rho_I R(x_o) F_1(p_o, k_o) F_2(p_o, h_o) \quad (4.23)$$

where $F_2(p_0, h_0)$ is defined by

$$F_2(p_0, h_0) = \frac{h_0 C(h_0) D_+(h_0) \hat{H}_-(h_0) J(y, h_0) \partial J(0, \alpha) / \partial \alpha \big|_{\alpha = p_0}}{p_0 J(0, h_0) C_+(h_0) \hat{H}_-(-p_0) J(y, p_0) D'_+(h_0) \left[\frac{D_-(p_0)}{C_-(p_0)} - \frac{C_+(p_0)}{D_+(p_0)} \right]} \quad (4.24)$$

However, from Eq. (4.21) it is seen that when $\text{Im } k_0 > \text{Im } p_0$ the scale range vanishes when $|x_0| \rightarrow \infty$. Therefore it is possible to conclude that the dominant mode is totally transmitted and no reflected component is observable. Even though p_0 characterizes the dominant mode in this case, any reflection must come from the k_0 mode. Provided the impedance of the ionosphere is sufficiently small and the source and the receiver are elevated and far from the shoreline and each other, the discontinuity is not observable to lowest order in the impedances. On the other hand, if the ionosphere impedance is not small, or if either the source or receiver is near the earth's surface, the portion of the incident wave reflected is proportional to the difference of the land and sea impedances.

4.3 The Recovery Effect

Strictly speaking, because of the complex impedances, every mode described by Eqs. (4.4-.7) is propagating, though those of higher order ($n > 0$) are strongly damped at low frequencies. At slightly higher frequencies, however, more than one mode may be sustained, particularly over the sea. At these higher frequencies, an electric field consisting of one dominant mode and an infinite number of heavily damped modes is incident upon the shoreline; the field transmitted beyond may consist of more than one observable mode. The energy from the suppressed modes over land would have been transferred to the higher order modes over the sea. This, in essence, is the "recovery effect" for VLF transmission.

Millington (1949) noted that the recovery effect was strongest when the observer was near the earth's surface. The same conclusion may be inferred from Section 4.2 where it was noted that reflection from the shoreline discontinuity was observed when the observer or the transmitter was near the earth. Therefore in order to illustrate the recovery effect, the observer is assumed to lie in a region near the earth and in a range $(\text{Im } k_1)^{-1} < |x| < (\text{Im } h_1)^{-1}$. Such a region is shown as shaded in Fig. 5. In this case, only one mode is

observable for $x > 0$ and two are observable for $x < 0$. Over the land ($x > 0$), the y-component of the electric field behaves like Eq. (4.10) while over the sea ($x < 0$) one has

$$E_y \sim \tau_I [e^{-ih_0 x} + F_3(h_0, h_1, k_1) e^{-ih_1 x}] \quad (4.25)$$

for $|x|$ in the prescribed range. The function $F_3(h_0, h_1, k_1)$ is given as

$$F_3(h_0, h_1, k_1) = \frac{h_1 J(y, h_1) C(h_1) D_+(h_1) \hat{H}_-(h_1)}{h_0 J(y_1, h_0) C(h_0) D_+(h_0) \hat{H}_-(h_0)} \quad (4.26)$$

$$\times \frac{J(0, h_0) D'(h_0) C_+(h_0)}{J(0, h_1) D'(h_1) C_+(h_1)}$$

Equation (4.26) may be written approximately as

$$F_3(h_0, h_1, k_1) \cong -ikb(Z_I + Z_L) \frac{h_1}{h_0} \frac{h_0 + k_1}{h_1 + k_1} \quad (4.27)$$

where quadratic terms in the impedances have been suppressed.

In order to compare the field on either side of the shoreline for the range $|x| > (\text{Im } k_1)^{-1}$ the squares of the amplitudes of the two cases are calculated and found to be

$$|E_y(x > 0)|^2 = e^{2\text{Im} k_0 x} + |\rho_I|^2 e^{-2\text{Im} k_0 x} \quad (4.28)$$

$$+ 2|\rho_I| \cos(2\text{Re} k_0 x + \arg \rho_I)$$

for an over-land observer and

$$\begin{aligned}
|E_y(x < 0)|^2 = |1+\rho_I|^2 \left\{ e^{2\text{Im}h_0x} + |F_3(h_0, h_1, k_1)|^2 e^{2\text{Im}h_1x} \right. \\
+ 2|F_3(h_0, h_1, k_1)|e^{\text{Im}(h_0+h_1)x} \cos[\text{Re}(h_0-h_1)x \\
\left. + \arg F_3(h_0, h_1, k_1)] \right\}
\end{aligned}
\tag{4.29}$$

for an over-sea observer. By comparing Eqs. (4.28) and (4.29) it can be seen from Eq. (4.27) that the effective increase in amplitude as the observer moves across the shore line is proportional to $|Z_I+Z_L|$. The recovery of $|E_y|^2$ is illustrated in Fig. 5. It should be noted that while $|E_y|^2$ is increasing as the observer moves from land to sea, $|E_x|^2$ is decreasing by an amount proportional to $|Z_I+Z_S|/|Z_I+Z_L|$. As a result, the wave vector may be thought of as "straightening up" as the wave passes from the relatively poor conducting land to the more highly conducting sea.

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Papers which do not directly apply to VLF mixed-path propagation, but relate to mixed-path propagation are listed below. Additional references may be found therein.

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Figure Captions

1. The earth-ionosphere duct showing schematic placement of transmitter T and receiver R.
2. The alpha-plane showing the occurrence of the $\pm k$ terms.
3. Path of integration for decomposition by sum of the $E(\alpha)/D_+(\alpha)C_-(\alpha)$ function
4. Evolution of $H_-(\alpha)$ by displacement of integral path to infinity in the upper-half z-plane.
5. Region of observation of the recovery effect so that one mode is observable for $x > 0$ but two are observable for $x < 0$.
6. The "recovery" of $|E_y|^2$ as the observer passes from land to sea.

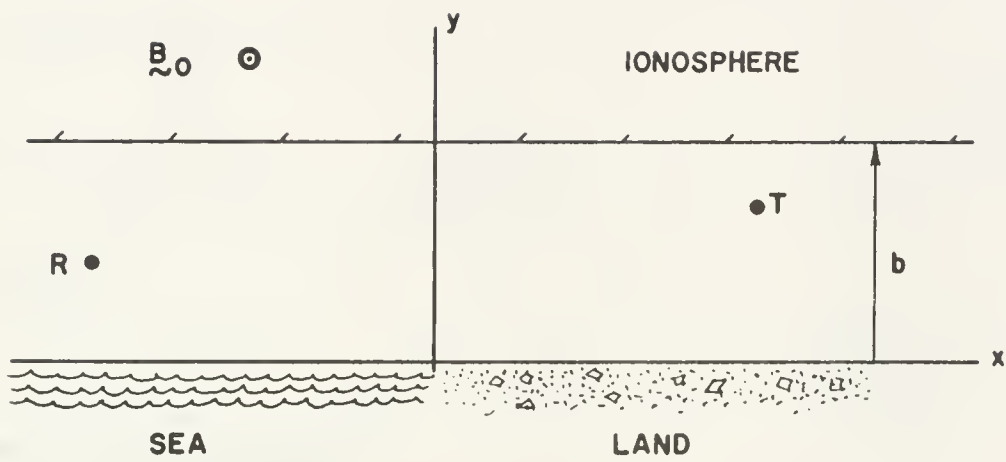


FIGURE 1

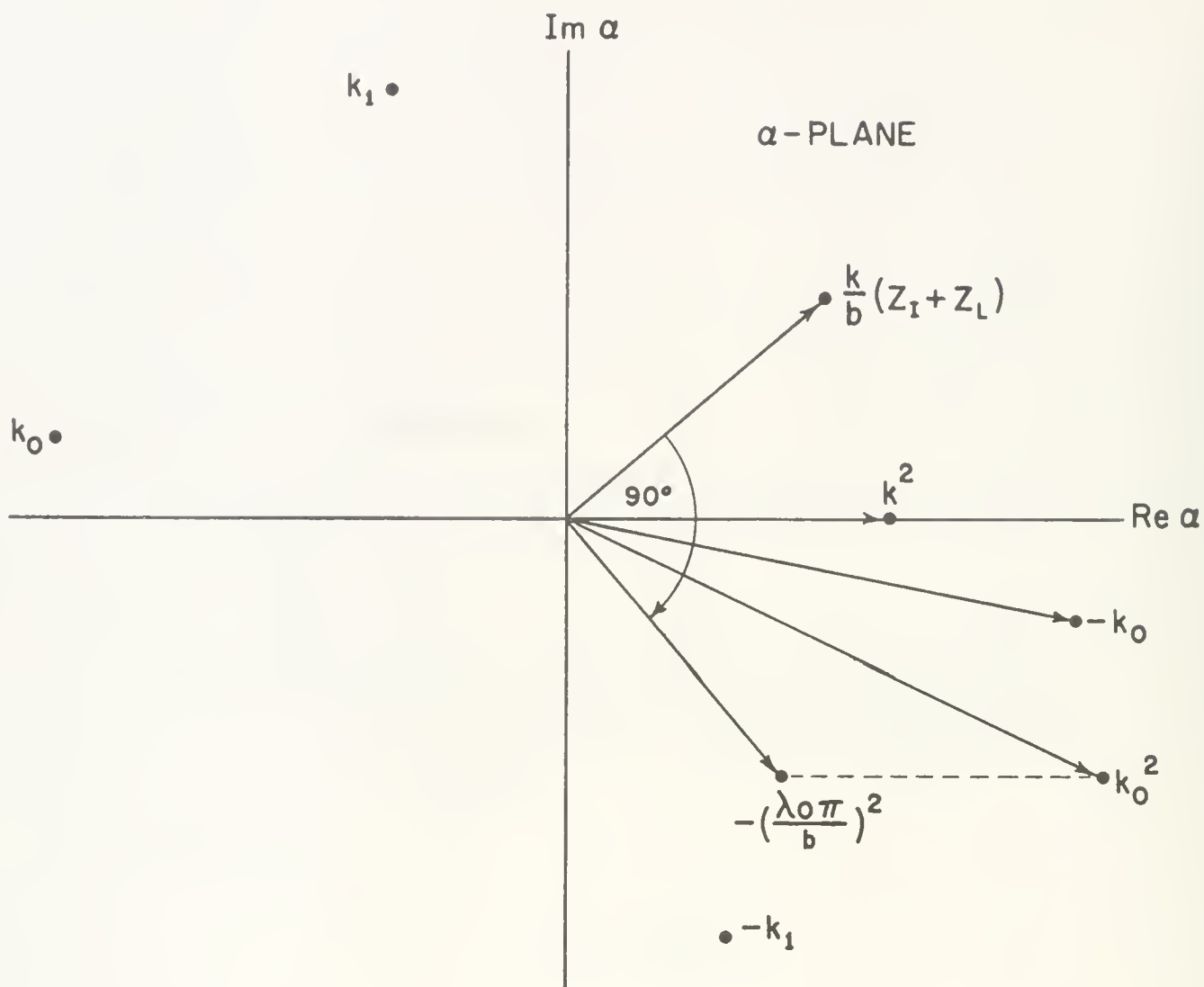


FIGURE 2

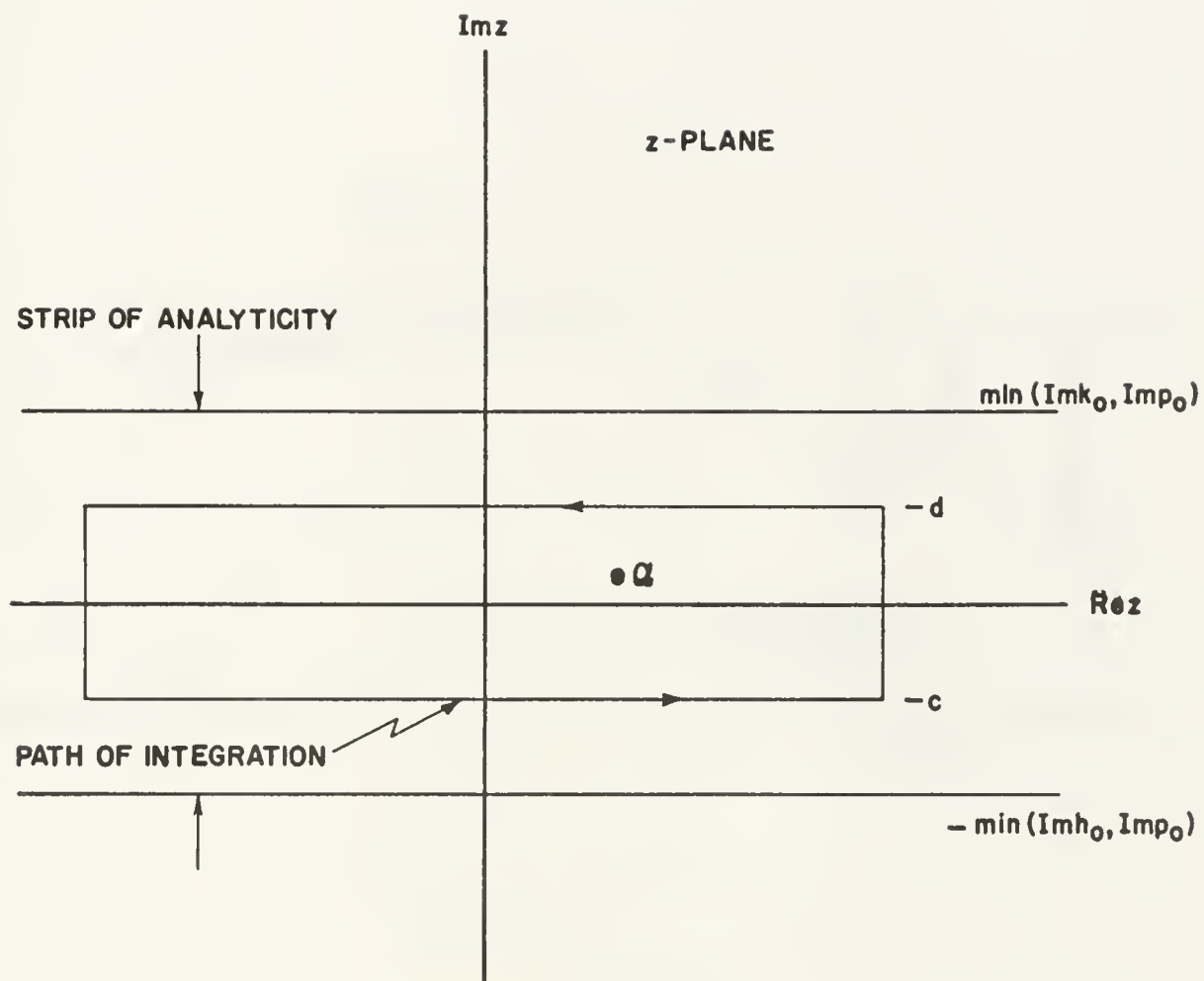


FIGURE 3

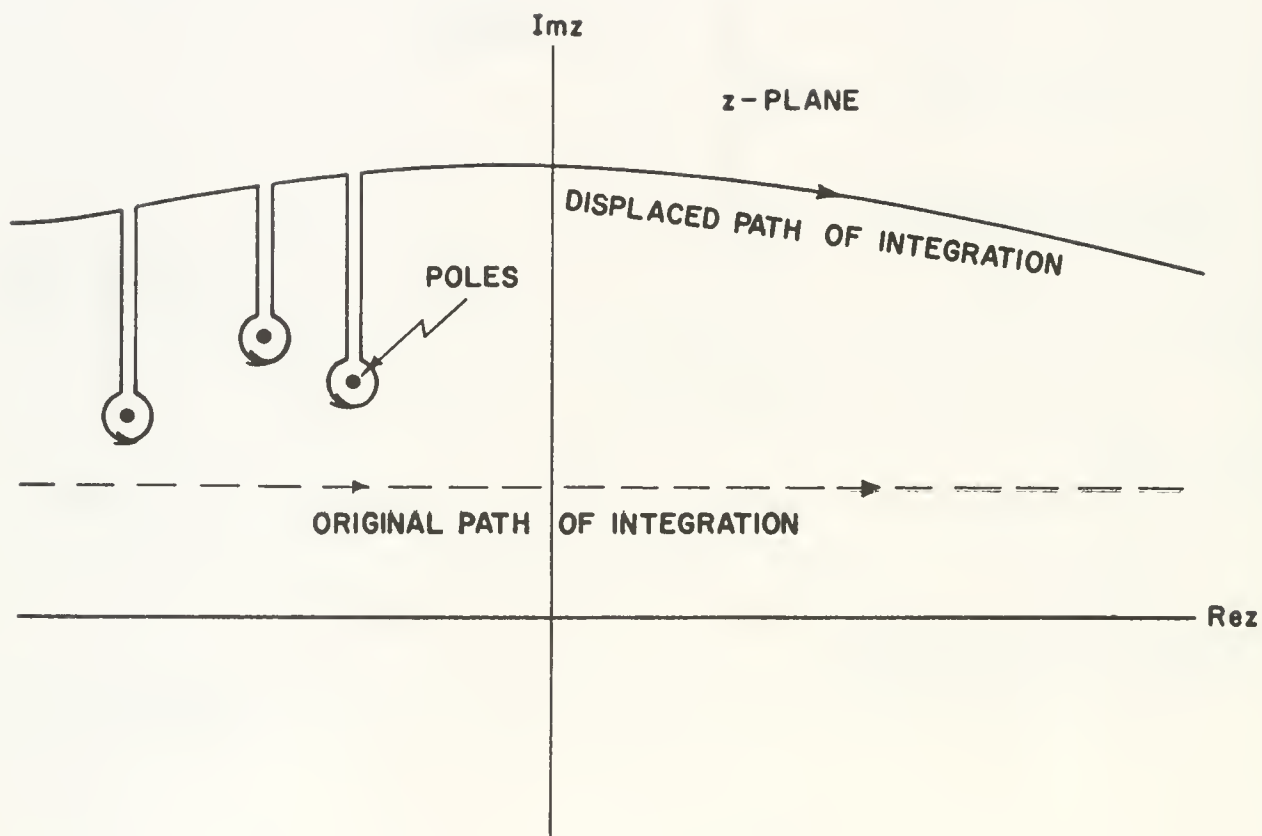


FIGURE 4

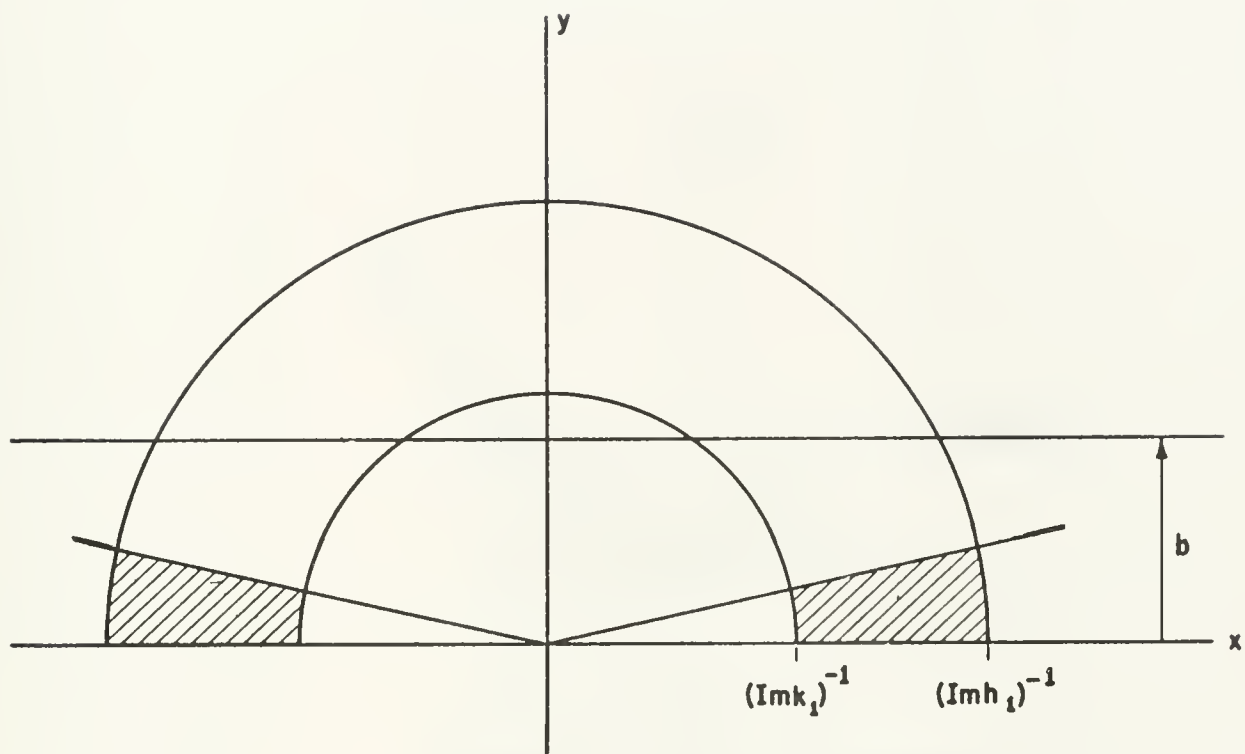


FIGURE 5

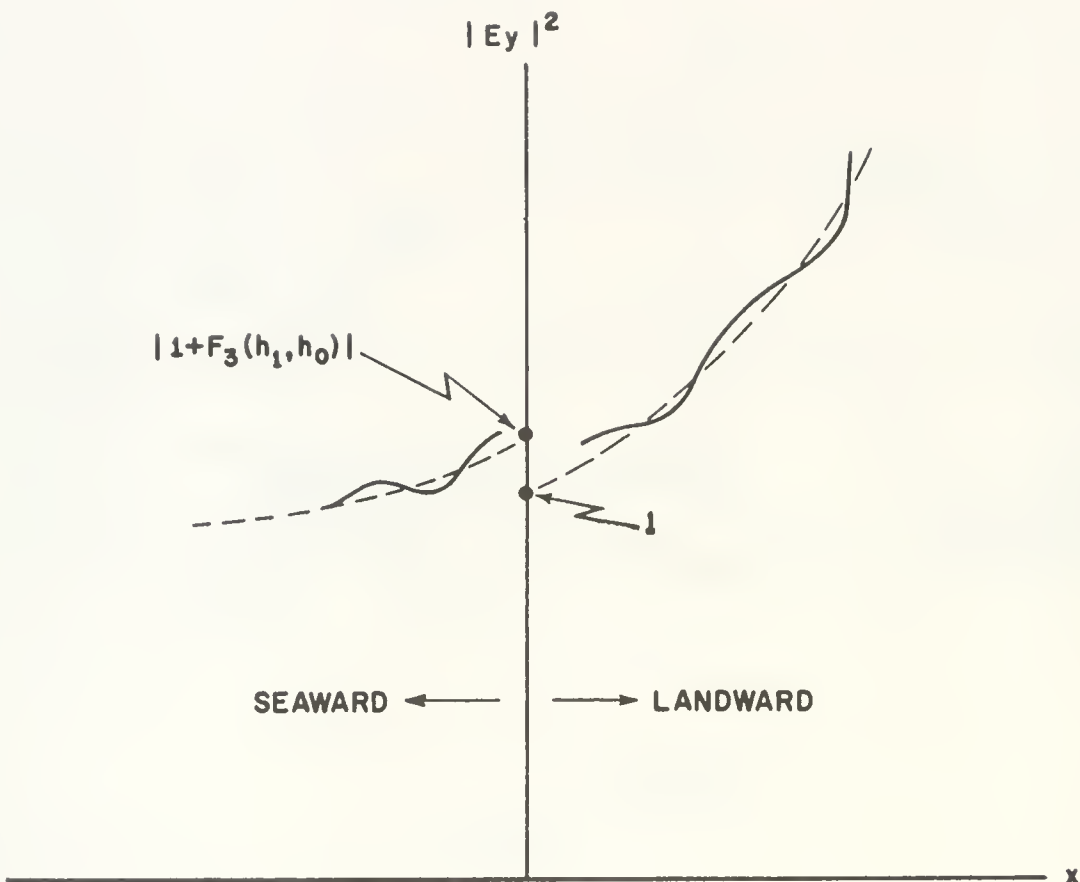


FIGURE 6

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Dobrott

Coastal refraction of
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